A STUDY OF GEOMETRIC OPTICS

CARMEN PARK, UNIVERSITY OF OREGON

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ADVISOR: JASON MURPHY

Abstract

While Maxwell's equations provide a complete description of light, analytical solutions are typically unavailable, and descriptions like computer rendering use simpler approximations like geometric optics to describe light. This approach however, fails to capture wave phenomena like diffraction and interference. This paper presents a detailed derivation that connects these two domains, and demonstrates that geometric optics is the high frequency asymptotic limit of electromagnetic wave theory. Following Kline [2] it begins with Maxwell's equations for a medium with a harmonic source and, applying a form of Duhamel's principle, constructs the general field solution. Through asymptotic expansion in the limit of large frequency the solution is separated into a leading-order "geometric optics" term governed by the eikonal equation and a series of "wave optics" corrections. This justifies the ray model as a high frequency approximation. As an alternative approach, we transition from wave fronts to rays, using a variational calculus approach based on Fermat's principle of least time. By rigorously formulating the action functional and its variation across a discontinuous interface, we derive the vector form of Snell's law as a necessary condition for path given a set of conditions. This work clarifies the foundational relationship between wave optics and geometric optics, confirming that ray modeling is a mathematically consistent limit of wave behavior and laying a foundation for a deeper exploration of wave optics approximations.

Contents

1 Introduction

A complete description of light as an electromagnetic wave is provided by Maxwell's equations. To solve any light-related problem, we must solve these equations with the appropriate initial and boundary conditions. One drawback is that exact analytical solutions to Maxwell's equations are only possible for a limited number of idealized scenarios, for example, a uniform plane wave in a vacuum or a homogeneous medium. Consequently, scientists and engineers, particularly in fields like high-frequency optics, often resort to the simpler approximations of geometric optics.

While highly useful, this geometric approach comes with a significant limitation: it fails to capture essential wave phenomena such as diffraction and interference, as it does not model the wave nature of light.

1.1 A Brief History

The foundations of geometric optics are centuries old. The law of reflection has been known since Euclid, and the law of refraction was formulated in the 17th century by René Descartes and Willebrord Snell. Robert Boyle and Robert Hooke observed interference; Olaf Rømer established the finite speed of light; Francesco Maria Grimaldi and Hooke discovered diffraction; Erasmus Bartholinus documented double refraction; and Isaac Newton discovered dispersion [3].

These empirical laws were later unified by Fermat's Principle of Least Time. It is an *ad-hoc* theory that minimizes the path length that light takes between two points. It is defined as the integral of the line of the index of refraction $n(x, y, z)$ along the path. A correct variational formulation states that the first variation of this integral must be zero.

The mathematical theory of geometrical optics is based on the work of William R. Hamilton. He introduced the concept of a characteristic function, which expresses the optical path length as a function of the endpoints of a light ray. The partial derivatives of this function yield the direction of the ray, which is used to solve problems involving lenses, mirrors, and anisotropic crystals. Hamilton's work also revealed the deep mathematical equivalence between Fermat's principle and Huygens' principle.

Geometric optics can describe most behaviors of light, however, it is unable to account for interference, diffraction, and polarization. In the early 19th century, experiments by Thomas Young, Augustin Fresnel, and others solidified the wave nature of light. Fresnel extended Huygens' principle by adding periodicity and creating a model that could explain diffraction. Most importantly, Young and Fresnel argued that light waves must be transverse, not longitudinal.

It was James Clerk Maxwell's synthesis of the laws of electromagnetism that led to the discovery of light as an electromagnetic wave. He demonstrated that the calculated speed of electromagnetic waves in a vacuum precisely matched the known speed of light. This critical finding established that the oscillating electric and magnetic fields, $\vec{E}$ and $\vec{H}$ respectively, in his equations are the physical quantities that propagate transversely as light. This insight provided a complete physical basis for wave optics and allows the laws of geometrical optics to be derived directly from these first principles.

1.2 Maxwell's Equations

This paper will trace this derivation, from the wave solution of Maxwell's equations through the eikonal equation, and demonstrate the derivation of Snell's law with a least action variational method.

Following the work of Kline [2], we apply a form of Duhamel's principle[1] to relate the electromagnetic field to an arbitrary charge distribution with harmonic time dependence. We begin with the macroscopic Maxwell equations in a medium, which form the foundation for this derivation:

$$\nabla \times \vec{H} - \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\frac{\partial \vec{F}}{\partial t} \tag{1}$$

$$\nabla \times \vec{E} + \frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} = 0, \tag{2}$$

where $\vec{E}, \vec{H} \in \mathbb{R}^3$ represent the electric and magnetic field vectors, respectively. The properties of the material are defined by the electric permittivity ϵ and the magnetic permeability μ, which are piecewise continuous functions of position (x, y, z).

The function $\vec{F}$ is assumed to be separable in space and time:

$$\vec{F}(\vec{r}, t) = \vec{g}(x, y, z)f(t), \quad \vec{r} = (x, y, z).$$

In this case, g can accommodate different sources. For example, in the case of a Hertzian dipole, $g = \vec{M}\delta(x, y, z)$ where $\vec{M}$ is the constant vector moment of the dipole, and $\delta(x, y, z)$ is the dirac delta function. The function above represents a source that is "on" at time $t = 0$. The corresponding initial conditions are as follows:

$$\vec{E}(\vec{r}, 0^-) = \vec{H}(\vec{r}, 0^-) = 0, \quad \text{and} \quad \vec{E}(\vec{r}, \infty) = \vec{H}(\vec{r}, \infty) = 0,$$

with $f(t) = 0$ for all $t < 0$. The harmonic time dependence is introduced by defining $f(t)$ for $t \geq 0$ as the piecewise wave equation:

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-i\omega t} & t \geq 0 \end{cases}$$

This choice of F, coupled with the zero initial conditions, allows the pulse solutions $\vec{E}_0$, $\vec{H}_0$ to be constructed using Duhamel's principle integral, transforming the problem of solving the inhomogeneous Maxwell equations into a more usable form.

[1] Duhamel's principle is a method for solving inhomogeneous differential equations. Kline attributes its specific application to Maxwell's equations to R. K. Luneburg, who derived it via Laplace transformations in a 1947–1948 lecture series at NYU.

Using the forcing function $f(t)$ and zero initial conditions allows the electromagnetic field solutions $\vec{E}$, $\vec{H}$ to be constructed from the pulse solutions $\vec{E}_0$, $\vec{H}_0$ with Duhamel's principle. The general solution is given by the convolution integral:

$$\vec{E}(t) = \frac{\partial}{\partial t} \int_0^t \vec{E}_0(t-\tau) f(\tau) d\tau, \tag{3}$$

$$\vec{H}(t) = \frac{\partial}{\partial t} \int_0^t \vec{H}_0(t-\tau) f(\tau) d\tau. \tag{4}$$

Here, τ is a placeholder variable for the convolution in time, representing the history of the source contribution.[2]

1.3 Transition to the Eikonal Equation and Wave Fronts

After confirming the solution's correctness via direct substitution, Kline's next objective is to interpret the integral. He achieves this by splitting it into two distinct physical components: a "geometric optics" part, which provides the main contribution, and a "wave optics" part, developed as a Taylor series via repeated integration by parts, which encapsulates the contributions from discontinuities and wave effects.

We begin with the expressions for the time-dependent fields of equations (3) and (4), using the change of variables $t - \tau = s$:

$$\vec{E}(t) = \frac{\partial}{\partial t} \left(\int_0^t E_0(s) e^{i\omega(t-s)} ds \right) \tag{5}$$

$$\vec{H}(t) = \frac{\partial}{\partial t} \left(\int_0^t H_0(s) e^{i\omega(t-s)} ds \right). \tag{6}$$

Kline notes the existence of a discontinuity hypersurface defined by $t = \frac{\psi(x,y,z)}{c}$, where the function ψ satisfies the conditions:

$$\nabla\psi \times [\vec{H}_0] = \epsilon[\vec{E}_0] = 0$$
$$\nabla\psi \times [\vec{E}_0] = \mu[\vec{H}_0] = 0,$$

and the ψ_α represent wave fronts satisfying the eikonal equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = \mu\epsilon$.

To analyze the high-frequency instances when $\omega \to \infty$, we integrate by parts. For the electric field, we define a spatial amplitude function $u(x, y, z)$ by evaluating the expression for $\vec{E}(t)$ and factoring out the time-harmonic part:

$$u(x, y, z) = E_0(t) - i\omega e^{-i\omega t} \int_0^t E_0(s) e^{i\omega s} ds.$$

[2] Source contribution simply means the total field at time t is the summation of all contributions from the source at previous times $\tau < t$.

Note that as $t \to \infty$, the field $E_0(t)$ approaches the static field $E_0(\infty)$, and the integral must converge sufficiently rapidly. A similar expression holds for the magnetic field:

$$v(x,y,z) = H_0(t) - i\omega e^{-i\omega t} \int_0^t H_0(s) e^{i\omega s} ds.$$

Thus, for large t, the full fields behave as $\vec{E}(t) \approx e^{-i\omega t} u(x,y,z)$ and $\vec{H}(t) \approx e^{-i\omega t} v(x,y,z)$, where u and v are time-independent spatial distributions representing the amplitude of the wave.

In the high-frequency limit, we extend the integrals to infinity and express u and v in terms of their asymptotic values:

$$u(x,y,z) = E_0(\infty) - i\omega \int_0^\infty \left(E_0(s) - E_0(\infty)\right) e^{i\omega s} ds, \tag{7}$$

$$v(x,y,z) = H_0(\infty) - i\omega \int_0^\infty \left(H_0(s) - H_0(\infty)\right) e^{i\omega s} ds. \tag{8}$$

At this point, we note that $E_0(s)$ is discontinuous at points s_α (for $\alpha = 1, \ldots, n$) corresponding to the wave fronts. Define the jumps as $[E_0]_\alpha = E_0(s_\alpha^+) - E_0(s_\alpha^-)$. We now integrate by parts to develop a series in powers of $1/\omega$:

$$\begin{aligned} u &= E_0(\infty) - i\omega \int_0^\infty \left(E_0(s) - E_0(\infty)\right) e^{i\omega s} ds \\ &= E_0(\infty) - \left(E_0(s) - E_0(\infty)\right) e^{i\omega s}\Big|_0^\infty + \int_0^\infty E_{0s}(s) e^{i\omega s} ds \\ &= E_0(0^+) + \int_0^\infty E_{0s}(s) e^{i\omega s} ds. \end{aligned}$$

Integrating by parts again accounts for the discontinuities in $E_{0s}(s)$:

$$\int_0^\infty E_{0s}(s) e^{i\omega s} ds = \frac{1}{i\omega} \left(\sum_\alpha [E_0]_\alpha e^{i\omega s_\alpha} - \int_0^\infty E_{0ss}(s) e^{i\omega s} ds \right).$$

Substituting back, the amplitude function becomes:

$$u = E_0(0^+) + \frac{1}{i\omega} \sum_\alpha [E_0]_\alpha e^{i\omega s_\alpha} - \frac{1}{i\omega} \int_0^\infty E_{0ss}(s) e^{i\omega s} ds. \tag{9}$$

Continuing to integrate (9) by parts provides a complete asymptotic expansion of the wave amplitude. The leading term is determined by the initial pulse, which, from our previous analysis, is given by $E_0(0^+) = \frac{-g}{\epsilon}$. The subsequent terms form a series in powers of $\frac{1}{(i\omega)}$

$$u = E_0(0^+) + \sum_{\alpha=1}^n [E_0]_\alpha e^{i\omega s_\alpha} + \frac{1}{i\omega} \sum_\alpha [E_{0s}]_\alpha e^{i\omega s_\alpha} + \cdots,$$

At this point, we can also acknowledge that s_a is synonymous with the eikonal equation ψ_α and the electric field and magnetic field have the same form like this:

$$u = E_0(0^+) + \sum_{\alpha=1}^{n} [E_0]_\alpha e^{i\omega\psi_\alpha} + \frac{1}{i\omega} \sum_{\alpha}^{n} [E_{0s}]_\alpha e^{i\omega\psi_\alpha} + \cdots, \tag{10}$$

$$v = H_0(0^+) + \sum_{\alpha=1}^{n} [H_0]_\alpha e^{i\omega\psi_\alpha} + \frac{1}{i\omega} \sum_{\alpha}^{n} [H_{0s}]_\alpha e^{i\omega\psi_\alpha} + \cdots. \tag{11}$$

This series nicely decomposes the solution into its physical components. The constant term $E_0(0^+)$, represents the geometric optics part, and defines the limiting high-frequency behavior where the wave propagates as a ray. The infinite series of terms proportional to $e^{i\omega s_\alpha}$ constitutes the wave optics correction. It describes wavelike phenomena, such as diffraction and the precise structure of those wave fronts. Each term in this series accounts for the effect of a discontinuity in the wavefront s_α with decreasing intensity as $\alpha \to \infty$.

Kline's method justifies the principles of geometric optics, since in high-frequency instances when $\omega \to \infty$, the wave optics corrections vanish, and the propagation of light is governed solely by the eikonal equation $ct = \psi(x, y, z)$. It is from this equation that fundamental laws like Snell's Law and the Law of Reflection can be directly derived.

2 Eikonal Equation deriving Geometric Optics

2.1 First Attempt

We start with a point source that emits spherical wave fronts. These are defined by the surfaces:

$$W_t = \{\vec{x} \in \mathbb{R}^3 : \psi(\vec{x}) = ct\},$$

where the function $\psi(\vec{x})$ is our eikonal (or phase function)[3] and $\vec{x} = (x_1, x_2, x_3)$.

Now we throw in a discontinuity: a flat planar interface at $x_2 = d$ where the index of refraction jumps:

$$n(\vec{x}) = \begin{cases} n_0 & x_2 \le d \\ n_1 & x_2 > d. \end{cases}$$

Here, n_0 and n_1 are just constants. We also pick some arbitrary point q sitting in the n_1 medium, so that $q_2 > d$.

Light rays travel in straight lines within each homogeneous medium, but their path bends at the discontinuity $x_2 = d$. Because of this, we model a single ray's trajectory in two segments. An initial segment from $t = 0$ to $t = T_*$ (the

[3] The eikonal ψ satisfies $|\nabla\psi(\vec{x})|^2 = n(\vec{x})^2$, where $n(\vec{x})$ is the index of refraction.

time of intersection) with direction $\vec{\omega}$, and a final segment from $t = T_*$ to $t = T$ with direction $\vec{\theta}$, terminating at an arbitrary point q.

$$x(t) = \begin{cases} \dfrac{c}{n_0}\omega t & 0 \leq t \leq T_* \\ q - \dfrac{c}{n_1}\theta(T-t) & T_* < t \leq T. \end{cases} \tag{12}$$

This also requires a couple of conditions. This considers the case where the ray bends into the second medium, so the angles are between 0 and 90 degrees, $\frac{\pi}{2} > \theta_2 > 0$ and the point q is definitely past the interface, $q_2 \geq d$.

The key idea is that at the precise moment $t = T_*$, the ray hits the interface. So, we impose the condition that the vertical component of its position must be d at that time. Let's use the expression for the path after the interface to describe this event. Plugging $t = T_*$ into the bottom part of the piecewise function gives:

$$x(T_*) = q - \frac{c}{n_1}\theta(T - T_*)$$

We want the y-component of this to be equal to d:

$$[x(T_*)]2 = q_2 - \frac{c}{n_1}\theta_2(T - T_*) = d.$$

Solving this equation for the unknown time T_* gives us the following,

$$T_* = T - \frac{n_1}{c\theta_2}(q_2 - d).$$

Finally, we make one last big assertion: the path has to be continuous. The limit of its position as we approach T_* from the left must equal the limit as we approach from the right:

$$\lim_{t \to T_*^-} x(t) = \lim_{t \to T_*^+} x(t).$$

Plugging in our piecewise definitions (12), this continuity condition leads to the following equation.

$$\frac{c}{n_0}\omega T_* = q - \frac{c}{n_1}\theta(T - T_*).$$

After a few different iterations of working with this equation and the previous one, nothing quite came together. There were too many variables $(T, T_*, q, d, \omega, \theta)$ and not enough constraints to make things cancel nicely. Having so many time variables as well in the equations made it difficult to isolate the relationship between the angles ω and θ that would give us Snell's law.

2.2 Least Action Principle

An alternative approach to describing light propagation, notably employed in an *ad-hoc* manner by Kline [3], utilizes the Principle of Least Action. Instead of tracking wave fronts, this method considers all possible paths a light ray could take and identifies the one that minimizes the action functional.

In order to take this approach, we define the least action principle as the transformation $\mathscr{L}(\gamma(s))$ and the integral

$$\int_0^1 n(\gamma(s))\|\gamma'(s)\|ds, \tag{13}$$

where $\|\gamma'(s)\|$ is the Euclidean norm of the tangent vector $\gamma'(s) = \frac{d\gamma}{ds}$ and n is the index of refraction in the medium. To find the path that minimizes this action, we introduce a smooth perturbation function $\phi(s)$ that vanishes at the endpoints, i.e., $\phi(0) = \phi(1) = 0$. The perturbed path is given by

$$\gamma_\epsilon(s) = \gamma(s) + \epsilon\phi(s). \tag{14}$$

The action along the path (14) becomes a function of ϵ:

$$\mathscr{L}(\gamma_\epsilon) = \int_0^1 n(\gamma_\epsilon(s))\|\gamma_\epsilon'(s)\|ds. \tag{15}$$

For γ to be a minimizing path, the derivative of this action with respect to ϵ must be zero at $\epsilon = 0$:

$$\begin{aligned}\frac{d}{d\epsilon}\mathscr{L}(\gamma_\epsilon)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\left(\int_0^1 n(\gamma_\epsilon(s))\,\|\gamma_\epsilon'(s)\|ds\right)\Big|_{\epsilon=0}\\ &= \int_0^1 \left(\nabla n(\gamma(s))\cdot\phi(s)\,\|\gamma'(s)\| + n(\gamma(s))\,\frac{\gamma'(s)\cdot\phi'(s)}{\|\gamma'(s)\|}\right)ds = 0.\end{aligned}$$

A complication arises when this variational method is applied across a discontinuity in the medium (e.g., at an interface between two materials). The standard formulation requires that the perturbation $\phi(s)$ be zero throughout the domain, including at the point of discontinuity. To properly account for the "jump" in the path at an interface, the integral in Eq. (15) must be split at the discontinuity point $s = t_0$.

$$\begin{aligned}\frac{d}{d\epsilon}\mathscr{L}(\gamma_\epsilon)\Big|_{\epsilon=0} = \Bigg(&\int_0^{t_0} n(\gamma(s)+\epsilon\phi(s))\|\gamma'(s)+\epsilon\phi'(s)\|ds+\\ &\int_{t_0}^1 n(\gamma(s)+\epsilon\phi(s))\|\gamma'(s)+\epsilon\phi'(s)\|ds\Bigg)\Big|_{\epsilon=0}.\end{aligned}$$

Since the parameter t_0 divides the index of refraction into two homogeneous mediums, it becomes a constant for either side and the equation simplifies to:

$$\frac{d}{d\epsilon}\mathscr{L}(\gamma_\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left(n_0\int_0^{t_0}\|\,\gamma\,(s)+\epsilon\,\phi\,(s)\|ds + n_1\int_{t_0}^1\|\,\gamma\,(s)+\epsilon\,\phi\,(s)\|ds\right)\Big|_{\epsilon=0},$$

and, after evaluating at $\epsilon = 0$,

$$= n_0 \int_0^{t_0} \frac{\gamma' \cdot \phi'}{\|\gamma'\|} ds + n_1 \int_0^{t_0} \frac{\gamma' \cdot \phi'}{\|\gamma'\|} ds. \tag{16}$$

At this point, we integrate by parts to make ϕ' return to ϕ so that all parts multiplied by it can be evaluated at zero as well as at the discontinuity t_0

$$\begin{aligned} =& n_0 \phi \cdot \frac{\gamma'}{\|\gamma'\|}\Big|_0^{t_0^-} - n_0 \int_0^{t_0} \frac{d}{ds}\left(\frac{\gamma'}{\|\gamma'\|}\right) \cdot \phi ds \\ +& n_1 \phi \cdot \frac{\gamma'}{\|\gamma'\|}\Big|_{t_0^+}^1 - n_1 \int_{t_0}^1 \frac{d}{ds}\left(\frac{\gamma'}{\|\gamma'\|}\right) \cdot \phi ds. \end{aligned}$$

The expression can be separated into terms that describe the interface and terms that describe the path within each medium:

$$\left[n_1 \frac{\gamma'}{\|\gamma'\|}\Big|_{t_0^+} - n_0 \frac{\gamma'}{\|\gamma'\|}\Big|_{t_0^-} \right] \cdot \phi(t_0) + \int_0^1 (\cdots) \cdot \phi ds = 0.$$

For this equation to hold for all smooth perturbations $\phi(s)$, both parts must individually be zero. This requires a further argument which can be found in appendix B.

The integral term vanishing implies that the path is a straight line in each medium. The boundary term vanishing implies the condition at the interface:

$$n_1 \frac{\gamma'}{\|\gamma'\|}\Big|_{t_0^+} - n_0 \frac{\gamma'}{\|\gamma'\|}\Big|_{t_0^-} = 0, \quad \text{or} \quad n_1 \hat{t}_1 - n_0 \hat{t}_0 = 0; \tag{17}$$

where $\hat{t} = \frac{\gamma'}{\|\gamma'\|}$ is the unit tangent vector. This is the vector form of Snell's law, governing the bending of light at a discontinuity.

3 Reflection

When I first contacted my advisor about this project, I expressed my interest in visual distortion and geometric optics. As a mathematics and computer science student passionate about graphics, my goal was to deepen my understanding of the mathematical models of light used in rendering.

While concepts like Fourier Transform, Dirac Delta distributions, and convolutions felt incredibly unfamiliar at the start, they began to make sense as mathematical tools to describe physical phenomena. Understanding equations (10) and (11) as an initial pulse and its following wave intensities was one of my favorite meetings. I hope to spend more time exploring the wave optics expansion to model diffraction. Similarly, seeing why the least action principle requires a weak solution formulation helped me better understand how rigorous mathematics can prove physical relationships.

Over these ten weeks, I gained a much better understanding of how light is modeled through the framework of how Maxwell equations inform geometric optics. While working with Jason I was also independently modeling the fundamental architecture of a ray-tracer. I followed the work of educators like Peter Shirley et. al. [5] and with a basis of a functional ray-tracer continued to implement modeling techniques from Pharr and Humphreys' "Physically Based Rendering - From Theory to Implementation" [4]. I documented this journey here [4].

Although Jos Stam was not the first researcher to implement wave optics in computer graphics, his 1999 paper "Diffraction Shaders" [6] was highly influential to my work. His mathematical simplifications appeared both realistically implementable and conceptually elegant. I plan to dedicate some of my time to exploring these techniques and implementing diffraction calculations in the future.

3.1 Acknowledgements

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I would also like to thank the University of Oregon Mathematics Department for this research opportunity. This experience has revealed the rich intersection of applied mathematics and computer graphics, providing an accessible entry point into the field. The work has been profoundly impactful, offering a foundation for further exploration in modeling, geometry, and computational graphics

A Fourier Analysis

The calculations above relate to monochromatic light for simplicity. However, since real light sources are never perfectly monochromatic, the Fourier transform is an essential tool for modeling and analyzing the broader spectra encountered in practical applications, as detailed by Goodman [1].

The Fourier transform (or frequency spectrum) of a complex-valued function g of three independent spatial variables $\vec{x} = (x, y, z)$ is represented as $\mathcal{F}(g)$ and is defined by the integral:

$$\mathcal{F}(g(\vec{\xi})) = \hat{f}(\vec{\xi}) = \iiint_{\mathbb{R}^3} g(\vec{x}) e^{-2\pi i \vec{\xi} \cdot \vec{x}} d\vec{x}, \tag{18}$$

where $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ is the spatial frequency vector. Its components ξ_1, ξ_2, ξ_3 are the spatial frequencies in the x, y, and z directions, respectively. The term $\vec{\xi} \cdot \vec{x}$ is the dot product $\xi_1 x + \xi_2 y + \xi_3 z$.

[4] In case you cannot view the hyperlink, it is ¡https://www.carmenpark.space/projects/ray-tracer/ray-tracer.html¿.

The corresponding inverse Fourier transform, which recovers the original spatial function from its frequency representation, is defined by:

$$\mathcal{F}^{-1}G(\vec{x}) = g(\vec{x}) = \iiint_{\mathbb{R}^3} G(\vec{\xi})e^{2\pi i\vec{\xi}\cdot\vec{x}}d\vec{\xi}.$$

B Weak Solution Principle

To define a "Weak Solution Principle", start with a smooth function F that depends on a path $\gamma(s)$ and its derivatives. We say that the equation $F(\gamma(s)) = 0$ for all $s \in (0,1)$ holds in a strong sense or in a pointwise sense. Say we only know that for every infinitely differentiable test function $\phi(s)$ that is zero at the boundaries (i.e. $\phi(0) = \phi(1) = 0$), then we get the following integral,

$$\int_0^1 \mathrm{F}\left(\gamma(s)\right)\phi\left(s\right)ds = 0,$$

and it follows that equation $F(\gamma(s)) = 0$ holds in the weak sense.

If the function is zero, then the integral against any test function ϕ must also be zero. We prove the converse; that if the integral is zero for all ϕ, then the function itself $F(\gamma(s))$ must be zero.

In order to prove this: We look at a specific point $s_0 \in (0,1)$, let $\phi_n(s) = n\,\phi\left(n(s-s_0)\right)$. Here, ϕ is a fixed function, it is a bump supported in the unit interval. then ϕ_n is supported where $|n(s-s_0)| \leq 1$ which implies $|s-s_0| \leq \frac{1}{n}$. This means ϕ_n is supported in a neighborhood $\frac{1}{n}$ around s_0. It has a height of n, width of $\frac{1}{n}$, and an integral of 1.

$$\int_{\mathbb{R}} \phi_n(s)ds = \int_{\mathbb{R}} n\phi(n(s-s_0))ds = \int_{\mathbb{R}} \phi(u)du = 1.$$

Since ϕ_n is converging to a delta function at s_0, ϕ_n converges as $n \to \infty$ to $\delta(s-s_0)$ and we can use this recover the value of $F(\gamma(s))$.

The "modified weak solution principle" then uses the same fundamental logic. Given $t_0 \in (0,1)$ and a path $\gamma(0,1) \to \mathbb{R}$, the following holds for all smooth these functions ϕ

$$\phi(t)G(\gamma(t)) + \int_0^1 F(\gamma(s)\phi(s)ds = 0.$$

We want to show that $G(\gamma(t_0)) = 0$ and $F(\gamma(s)) = 0\ \forall s \in (0,1)$. In order to reach this conclusion, we apply the weak solution principle (WSP).

First, to show $F(\gamma(s)) = 0$, consider the test functions ϕ that are zero at t_0. For these, the local term vanishes and is reduced to $\int_0^1 \phi(s)F(\gamma(s)) = 0$. By the WSP this implies $F(\gamma(s) = 0 \quad \forall s \in (0,1)$.

Second, we show that $G(\gamma(t_0)) = 0$. Since $F = 0$, It is known that $\phi(t_0)G(\gamma(t_0)) = 0 \quad \forall \phi$, and it then follows that $G(\gamma(t_0)) = 0$.

This form is similar to the results obtained in Section 2.2 after integration by parts, where the vanishing of the integral term constrains the function F (e.g., making it piecewise linear), and the vanishing of the local term G enforces a boundary condition relating velocity vectors to the index of refraction.

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